Problem 2

Suppose the rod has a constant internal heat source, so that the basic equation describing the heat flow within the rod is

$$u_t = \alpha^2 u_{xx} + 1 \qquad 0 < x < 1$$

Suppose we fix the boundaries' temperatures by $u(0,t) = 0$ and $u(1,t) = 1$. What is the steady-state temperature of the rod? In other words, does the temperature $u(x,t)$ converge to a constant temperature $U(x)$ independent of time?
HINT Set $u_t = 0$. It would be useful to graph this temperature. Also start with an initial temperature of zero and draw some temperature profiles.

Solution

Take advantage of the fact that the heat equation is linear: Assume the solution for the temperature $u(x,t)$ has a steady component and a transient component.

$$u(x,t) = R(x) + U(x,t)$$

Substitute this formula into the PDE.

$$\frac{\partial}{\partial t}[R(x) + U(x,t)] = \alpha^2 \frac{\partial^2}{\partial x^2}[R(x) + U(x,t)] + 1$$

Evaluate the derivatives.

$$U_t = \alpha^2 [R''(x) + U_{xx}] + 1$$

If we set

$$\alpha^2 R''(x) + 1 = 0,$$

then the previous equation becomes

$$U_t = \alpha^2 U_{xx}.$$

Now substitute the formula for $u(x,t)$ into the boundary conditions.

$$\begin{aligned} u(0,t) &= 0 & \rightarrow \quad & R(0) + U(0,t) = 0 \\ u(1,t) &= 1 & \rightarrow \quad & R(1) + U(1,t) = 1 \end{aligned}$$

Set $R(0) = 0$ and $R(1) = 1$ so that $U(0,t) = 0$ and $U(1,t) = 0$. Solve the ODE for the steady-state temperature by integration.

$$R''(x) = -\frac{1}{\alpha^2} \quad \rightarrow \quad R'(x) = -\frac{1}{\alpha^2}x + C_1 \quad \rightarrow \quad R(x) = -\frac{1}{2\alpha^2}x^2 + C_1 x + C_2$$

Apply the two boundary condtions to determine C_1 and C_2.

$$R(0) = C_2 = 0$$

$$R(1) = -\frac{1}{2\alpha^2} + C_1 + C_2 = 1$$

Solving this system yields $C_1 = 1 + 1/(2\alpha^2)$ and $C_2 = 0$. Therefore, the steady-state temperature is

$$R(x) = -\frac{1}{2\alpha^2}x^2 + \left(1 + \frac{1}{2\alpha^2}\right)x.$$